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Stability Properties of Autonomous Homogeneous Polynomial Differential Systems

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A geometrical approach is used to derive a generalized characteristic value problem for dynamic systems described by homogeneous polynomials. It is shown that a nonlinear homogeneous polynomial system possesses eigenvectors and eigenvalues, quantities normally associated with a linear system. These quantities are then employed in studying stability properties. The necessary and sufficient conditions for all forms of stabilities characteristic of a two-dimensional system are provided. This result, together with the classical theorem of Frommer, completes a stability analysis for a two-dimensional homogeneous polynomial system.

INTRODUCTION

Consider an autonomous homogeneous polynomial differential equation, or system

$$\Sigma_X: \quad \dot{X} = F[X],$$

where $X \in \mathbb{R}^m$, and $F[X] = [f_1(X),\ldots,f_m(X)]^t$, with each $f_i(X)$ being a real homogeneous polynomial, or a form of degree $n \geqslant 1$. When $n = 1$, Σ_X represents a linear system, (i.e., for this case it is usual to write $F[X] = AX$, where A is an $m \times m$ real matrix). For $n > 1$, Σ_X is nonlinear.

A main objective of this paper is to study stability properties of Σ_X. It is shown that the existing relations between the qualitative behavior of a linear system and its eigenvalues can be generalized for the case $n > 1$. Thus, in the first part of the paper a generalized characteristic value problem, abbreviated as GCVP, is formulated. This problem is a subclass of a nonlinear eigenvalue problem studied in [15, 18].

To treat the GCVP results in a compact form, the tools of algebraic geometry are utilized. Results are then used to construct characteristic solutions and examine qualitative properties of Σ_X. The paper is concluded by establishing a necessary and a sufficient condition for asymptotic stability in the large for a two-dimensional Σ_X. This result together with Frommer's Theorem, [4, 5], completes a stability analysis for a two-dimensional case.

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GCVP AND PROJECTION EQUATION

The GCVP for an autonomous homogeneous polynomial differential equation is defined as

$$F[S] - \lambda S = 0, \tag{1}$$

where $\lambda \in \mathbb{C}$ is called the eignenvalue, and $S \in \mathbb{C}^m$ is an associated eigenvector. Every eigenvector S is determined by the set of homogeneous coordinates $(s_1,..., s_m)$. For $n = 1$, it is customary to write Eq. (1) as

$$[A - \lambda I] S = 0, \tag{2}$$

where I is the $m \times m$ identity matrix. A "standard" way of solving Eq. (2), is to evaluate λ via a characteristic equation

$$\det\{A - \lambda I\} = 0, \tag{3}$$

and then to realize S through the set of linear algebraic equations. Unfortunately, when $n > 1$ a convenient representation like (3) does not exist. Thus a different technique for solving a GCVP must be introduced.

Let $\mathbb{K}$ be $\mathbb{C}$ or $\mathbb{R}$, and let $X \in \mathbb{K}^m$. Then for any nonzero coordinate $x_i \in X$, there exists a projection vector $Z \in \mathbb{K}^m$, such that

$$X = x_i Z. \tag{4}$$

It is clear that Z is determined by the set of homogenous coordinates $(z_1,..., z_{i-1}, 1, z_{i+1},..., z_m)$. A particular x_i in Eq. (4) is called a primal state or a primal coordinate.

If now the transformation (4) is substituted in Σ_X, then

$$F[x_i Z] = f_i(x_i Z)Z + x_i \dot{Z}. \tag{5}$$

Since the components of $F[X]$ are all forms of degree n, the last equation becomes

$$\dot{Z} = x_i^{n-1}(F[Z] - f_i(Z)Z). \tag{6}$$

This auxiliary equation is a projection equation. The solutions of a projection equation are the nonhomogeneous coordinates, or states of Z with respect to a primal state x_i, namely $(z_1,..., z_{i-1}, z_{i+1},..., z_m)$. Equation (6) is also observed by Coleman in [2].

In "general," any $X \in \mathbb{C}^m$ can have up to m nonzero coordinates. This implies that Z has up to m equivalent representations. Let

$C(Z) = [Z^j = (1/z_j)Z;$ for all $z_j \neq 0,\ z_j \in Z]$ be the set of all equivalent projection vectors of Z. Now, for any $Z^j \in C(Z)$, the equation

$$\dot{Z}^j = x_j^{n-1}(F[Z^j] - f_j(Z^j)Z^j) \tag{7}$$

is equivalent to the projection Eq. (6). Thus there could be at most m equivalent projection equations.

PROPOSITION 1. All *nontrivial singular or critical solutions of* all *projection equations determine solutions of a GCVP.*

Proof. The singular solutions of Eq. (6) are given by

$$F[Z] - f_i(Z)Z = 0. \tag{8}$$

This system of polynomial algebraic equations of degree $\leqslant n+1$ is referred to as a pencil equation. In order to consider all nontrivial solutions of a pencil equation, the components of Z must belong to an algebraically closed field $\mathbb{C}$, implying $Z \in \mathbb{C}^m$.

Now, let $S \in \mathbb{C}^m$ be the solution of Eq. (8), and let

$$\lambda = f_i(S) \tag{9}$$

from which Eq. (1) follows.

The form $f_i(X)$, in Eq. (9), is called a primal element of $F[X]$. Observe that GCVP implies $\lambda = S^T F[S]/\|S\|^2$.

The results of Proposition 1 are identical to those obtained by nonassociative algebra in [3, 12, 16, 17]. In terms of this algebra, the eigenvectors with non-zero (zero) eigenvalues are known as idempotents (nilpotents). Further, the eigenvalues of a GCVP are related to the radial type numbers defined in [2].

PROPOSITION 2. *Let $S \in \mathbb{K}^m$, with associated $\lambda \in \mathbb{K}$, satisfy the* GCVP. *Then for any $c \in \mathbb{K}$, cS is an equivalent eigenvector with associated eigenvalue*

$$\lambda_e = \lambda c^{n-1}. \tag{10}$$

Proof. Let $c \in \mathbb{K}$, and multiply Eq. (1) by c^n to obtain

$$F[cS] - (\lambda c^{n-1})cS = 0, \tag{11}$$

from which the result follows.

The immediate consequence of Proposition 2 is that every eigenvector equivalence class $C(S)$ has an associated eigenvalue equivalence class $\mathscr{E}(\lambda, S) = \{\lambda^j = \lambda/(s_j^{n-1});$ for all $s_j \neq 0,\ s_j \in S\}$. Further, observe that every

$\lambda^j \in \mathscr{E}(\lambda, S)$ is related to $S^j \in C(S)$ via a primal element $f_j(X)$ by the expression

$$\lambda^j = f_j(S^j). \tag{12}$$

The extensions of equivalence classes $C(S)$ and $\mathscr{E}(\lambda, S)$ are defined by the sets $\bar{C}(S) = \{cS; c \in \mathbb{K}\backslash 0, S \in \mathbb{K}^m\}$ and $\bar{\mathscr{E}}(\lambda, S) = \{\lambda c^{n-1}; c \in \mathbb{K}\backslash 0, \lambda = f_i(S)$, where $f_i(X)$ is a form of degree $n\}$. Now, let $D(\mathbb{K}^m) = \{S \in \mathbb{K}^m$; all nonequivalent eigenvectors satisfying GCVP$\}$, and $\Lambda(\mathbb{K}) = \{\lambda \in \mathbb{K}; \lambda = f_i(S)$, for all $S \in D(\mathbb{K}^m)\}$ be the respective nonequivalent eigenvector and eigenvalue sets. The extension of the last two sets is given by $\bar{D}(\mathbb{K}^m) = \{\bigcup_i \bar{C}(S_i)$; for all $S_i \in D(\mathbb{K}^m)\}$ and $\bar{\Lambda}(\mathbb{K}) = \{\bigcup_i \bar{\mathscr{E}}(\lambda_i, S_i)$; for all $\lambda_i \in \Lambda(\mathbb{K})$ and $S_i \in D(\mathbb{K}^m)\}$.

Remark. When $n = 1$, $\mathscr{E}(\lambda, S)$ and its extension are singletons.

Algebraic Geometry Interpretation

A reader not familiar with the language of algebraic geometry is referred to [6, 10]. Let $\mathbb{R}[x_1,..., x_m]$ be a ring of polynomials with coefficients in $\mathbb{R}$ and indeterminates $x_1,..., x_m$. If now $\mathscr{S}$ is any set of polynomials in $\mathbb{R}[x_1,..., x_m]$, then $V(\mathscr{S})$ represents an affine algebraic set. Further, let $P^{m-1}(\mathbb{K})$ be a projective $(m-1)$-space over $\mathbb{K}$. Then $V(\mathscr{S})$ is also an algebraic set in $P^{m-1}(\mathbb{K})$, or a projective algebraic set.

PROPOSITION 3. *Let* $\mathscr{S}_i = \{f_1(Z) - f_i(Z)z_1,..., f_m(Z) - f_i(Z)z_m\}$ *be the set of polynomials characterizing a pencil equation defined with respect to a primal state* x_i. *Define* $\mathscr{S}_t = \{\bigcup_{i=1}^m \mathscr{S}_i\}$. *Then it follows that*:

$$\text{(i)} \quad \bar{D}(\mathbb{K}^m) \equiv V(\mathscr{S}_t) \tag{13}$$

$$\text{(ii)} \quad \bar{D}(\mathbb{K}^m) \textit{ is a projective algebraic set.} \tag{14}$$

Proof. Part (i) is implied by the definition of $V(\mathscr{S}_t)$. Part (ii) follows from the fact that each $\bar{C}(S_i) \subseteq \bar{D}(\mathbb{K}^m)$ forms a $\mathbb{K}$-line through the origin of $\mathbb{K}^m$.

The statement ii) in the last proposition implies that Σ_X has a pencil type trajectory through the origin of $\mathbb{R}^m$ whenever a projective algebraic set $\bar{D}(\mathbb{R}^m)$ is nonempty. This result is easily observed when examining the phase portraits of two-dimensional quadratic systems, given in [1, 11, 12, 14, 20].

A point $S \in \bar{D}(\mathbb{K}^m)$ is called simple if the Jacobian of $F[X] - f_j(X)X$, evaluated at $X = S^j$, $S^j \in C(S)$, is not zero. Otherwise, a point is called multiple or singular. Every point $S \in \bar{D}(\mathbb{K}^m)$ has an associated multiplicity σ_s. S is simple if and only if $\sigma_s = 1$.

PROPOSITION 4. *The eigenvector multiplicity implies the eigenvalue multiplicity, but the reverse is not true.*

Proof. Clearly if $S \in \bar{D}(\mathbb{K}^m)$ has a multiplicity σ_s, then the relation (9) implies at least the same multiplicity of λ. However, two or more simple or multiplicative nonequivalent eigenvectors can have identical eigenvalues.

Any $f_i(X) \in F[X]$ is said to be x_i-decomposable if

$$f_i(X) = x_i g_i(X), \tag{15}$$

where $g_i(X)$ is a form of degree $n-1$. If every element of $F[X]$ is x_i-decomposable then $F[X]$ is X-decomposable. Σ_X is said to be trivial if $F[X]$ is X-decomposable, and

$$g_1(X) = \cdots = g_m(X) \equiv g(X). \tag{16}$$

PROPOSITION 5. All *eigenvectors of a trivial Σ_X are elements of $\bar{D}(\mathbb{R}^m)$. Moreover, $\bar{D}(\mathbb{R}^m) \equiv P^{m-1}(\mathbb{R})$.*

Proof. For a trivial Σ_X, a GCVP is given by

$$[g(S) - \lambda] S = 0. \tag{17}$$

Let $\lambda = g(S)$. Then for any $S \in \mathbb{R}^m$ the last equation is satisfied. This implies $\bar{D}(\mathbb{R}^m)$ contains *all* lines through the origin of $\mathbb{R}^m$.

PROPOSITION 6. *A two-dimensional Σ_X of degree n has either infinitely many nonequivalent eigenvectors, all belonging to $D(\mathbb{R}^2)$, or there are finitely many nonequivalent elements in $D(\mathbb{K}^2)$ with the sum of multiplicities being exactly $n+1$.*

Proof. The Proposition 5 implies that $D(\mathbb{R}^2)$ has inifinitely many elements whenever Σ_X is trivial. Now, let Σ_X be nontrivial of degree n. Then a pencil equation with $f_1(X)$ as a primal element becomes

$$\begin{aligned} (1 - s_1) f_1(S) &= 0, \\ f_2(S) - s_2 f_1(S) &= 0. \end{aligned} \tag{18}$$

The first equation is trivially satisfied with $s_1 = 1$, and the second becomes a polynomial in s_2 of degree at most $n+1$. If the second polynomial has a degree $k < n+1$, then this indicates the presence of a solution at infinity with a multiplicity $(n+1-k)$. To explicitly evaluate a solution at infinity, a pencil equation with a primal element $f_2(X)$ must be considered. In any event $n+1$ solutions, including multiplicities, always exist.

Proposition 6 in general cannot be extended to the higher dimensional cases. The exception is when Σ_X is linear. In this case, the number of non-equivalent eigenvectors is always known, [7]. However, when $n > 1$, techniques for determining an algebraic set $D(\mathbb{K}^m)$ become very involved.

Characteristic Solutions

The solutions evolving on manifolds in $\mathbb{R}^m$, determined by $\bar{D}(\mathbb{K}^m)$, are referred to as characteristic solutions. In general, it is not possible to evaluate closed form characteristic solutions for complex eigenvectors. Fortunately, characteristic solutions for real eigenvectors are given by the following proposition.

PROPOSITION 7. *If Σ_X, of degree $n \geqslant 1$, has a non-empty algebraic set $\bar{D}(\mathbb{R}^m)$, then for any $S \in \bar{D}(\mathbb{R}^m)$ a characteristic solution is defined by*:

(a) *for $n = 1$*,

$$X(t) = x_{jo} \exp\{\lambda^j (t - t_0)\}\, S^j,$$

(b) *for $n > 1$*,

$$X(t) = \frac{x_{jo}}{[1 - x_{jo}^{n-1}(n-1)\,\lambda^j (t - t_0)]^{1/n-1}}\, S^j,$$

where $S^j \in C(S)$, $\lambda^j = f_j(S^j) \in \mathscr{E}(\lambda, S)$ and $x_{jo} S^j = X(t = t_0)$.

Proof. Substitute Eq. a) and b) into Σ_X, and the result follows. The result comparable to characteristic solution b) is found in the proof of Theorem 9 in [3].

Qualitative Results

When studying stability properties, it is important to recognize a fundamental topological property of Σ_X, namely:

PROPOSITION 8. *Either Σ_X has an isolated singular point at the origin, or there exists a pencil on which all points are singular.*

Proof. Trivially, $X \equiv 0$ is a singular point. Next, suppose $X^* \not\equiv 0$ is another singular point. Then by definition $F[X^*] = 0$. But for any $r \in \mathbb{R}$,

$$r^n F[X^*] \equiv F[X^{**}] = 0, \tag{19}$$

where $X^{**} = rX^*$, which represents the equation of a line parameterized by r.

Using Propositions 7 and 8, one obtains a relation between the singular point and the eigenvalues:

PROPOSITION 9. *Σ_X has a pencil on which all points are singular if and only if there exists an eigenvector $S \in D(\mathbb{R}^m)$ such that its associated eigenvalue $\lambda \in \Lambda(\mathbb{R})$ is 0.*

Proof. The "if" part follows by examination of the characteristic solutions (a) and (b). To show the sufficiency, suppose that Σ_X has a line through the origin on which all points are singular. Then by the proof of statement (ii), Proposition 3, there exists an extended equivalence class $\bar{C}(S) \in \bar{D}(\mathbb{R}^m)$. Now, since every point on the $\mathbb{R}$-line $\bar{C}(S)$ is singular, the characteristic solution evolving on the pencil must equal the initial value for all t. This condition is achieved when λ^j is zero, implying every element of $\bar{\mathscr{E}}(\lambda, S)$ is zero.

The types of stability treated are asymptotic stability in the large, marginal stability, and periodic stability. For the precise definitions of these properties see [9, 13, 19].

In [8], it is shown that Σ_X is never asymptotically stable in the large when the degree of homogeneous polynomials is even. This property is observed in the characteristic solution (b) (i.e., when n is even, an initial condition x_{j_0} can always be selected such that the solution explodes). However, for n odd, one has the following result:

PROPOSITION 10. *If Σ_X, with n odd, has a nonempty set $D(\mathbb{R}^m)$, then a necessary condition for asymptotic stability in the large is that for every $S \in D(\mathbb{R}^m)$ the associated $\lambda \in \Lambda(\mathbb{R})$ is negative.*

Proof. First, suppose there exists $S \in D(\mathbb{R}^m)$ with an associated eigenvalue $\lambda > 0$. Then the characteristic solution along the $\mathbb{R}$-line given by $\bar{C}(S)$ has the following properties:

(a) For $n = 1$, $\|X(t)\| \rightarrow_{t\rightarrow\infty} \infty$.

(b) For $n > 1$, $\|X(t)\| \rightarrow_{t\rightarrow t_e} \infty$, where

$$t_e = \frac{1}{\lambda^j X_{jo}^{n-1}(n-1)} + t_o, \tag{20}$$

and $\lambda^j \in \mathscr{E}(\lambda, S)$.

But these properties contradict a condition for asymptotic stability in the large.

Next, let there be $S \in D(\mathbb{R}^m)$ for which $\lambda = 0$. Then by Proposition 9, the $\mathbb{R}$-line given by $\bar{C}(S)$ contains only singular points. Thus the condition for asymptotic stability in the large is violated again.

Finally, if every $S \in D(\mathbb{R}^m)$ has an eigenvalue $\lambda < 0$, then the characteristic solution evolving on the $\mathbb{R}$-line, defined by $\bar{C}(S)$, has the property $\|X(t)\| \to_{t \to \infty} 0$.

Remark. In the case when Σ_X has an odd degree n, the extended eigenvalue class $\bar{\mathscr{E}}(\lambda, S)$ is sign-preserving. That is, λ is negative if and only if any element in $\bar{\mathscr{E}}(\lambda, S)$ is negative.

Using the last Remark, Proposition 10 can be stated as:

PROPOSITION 10a. *If Σ_X has an odd degree n and a nonempty set $D(\mathbb{R}^m)$, then a necessary condition for asymptotic stability in the large is that every $\bar{C}(S) \in \bar{D}(\mathbb{R}^m)$ has an associated $\bar{\mathscr{E}}(\lambda, S) \in \bar{\Lambda}(\mathbb{R})$ in which any element is negative.*

The result of Proposition 10, or 10a, to a large extent resembles that of Theorem 2 in [2]. There, Coleman uses the sign of the radial type number in order to establish whether the trivial solution, $x = 0$, is asymptotically stable.

A type of stability applying to Σ_X with n even or odd is the marginal stability.

PROPOSITION 11. *If Σ_X has a nonempty set $D(\mathbb{R}^m)$, then the necessary conditions for stability, or marginal stability, are:*

(i) *For n* even, *any $\lambda \in \bar{\Lambda}(\mathbb{R})$ associated with $S \in \bar{D}(\mathbb{R}^m)$ is* 0.

(ii) *For n* odd, *any $\lambda \in \bar{\Lambda}(\mathbb{R})$ associated with $S \in \bar{D}(\mathbb{R}^m)$ is $\leqslant 0$.*

The proof follows from the inspection of the characteristic solutions (a) and (b).

The last two propositions heavily depend on the fact that $D(\mathbb{R}^m)$ or $\bar{D}(\mathbb{R}^m)$ is non-empty. This is because, in general, there are no known closed-form characteristic solutions when eigenvectors are complex. Further, sufficiency conditions are missing in each of the propositions, making the results weak. In the next section of two-dimensional Σ_X is studied, and sufficiency conditions for three types of stability are provided.

Two-Dimensional Σ_X

PROPOSITION 12. *Consider a two-dimensional odd Σ_X for which the set $D(\mathbb{R}^2)$ is non-empty. Then a necessary and sufficient condition for asymptotic stability in the large is that every $\lambda \in \Lambda(\mathbb{R})$ associated with $S \in D(\mathbb{R}^2)$ is negative.*

Proof. The necessary condition is implied by Proposition 10. Now, the sufficiency condition follows from the fact that Σ_X possesses an asymptote, or asymptotes, defined by the elements of $\bar{D}(\mathbb{R}^2)$. Thus the solution evolving on any integral curve of Σ_X approaches the characteristic solution as $t \to t_e$, $t_e \leqslant \infty$.

From the last proposition it follows that, as long as $D(\mathbb{R}^2)$ is nonempty, the eigenvalues corresponding to $S \in D(\mathbb{C}^2)$ will not affect the stability properties of Σ_X. However, when $D(\mathbb{R}^2)$ is empty, the extension of Frommer's Theorem [4, 5] gives the following elegant property.

PROPOSITION 13. *When an odd Σ_X has an empty set $D(\mathbb{R}^2)$, define an oriented line integral*

$$I_Z = \int_{\mathbb{R}} \frac{f_1(Z^*)}{f_2(Z^*) - z_2 f_1(Z^*)} \, dz_2, \tag{21}$$

where $Z^* = Z \mid z_{1=1}$, and integration is taken along all of the z_2 projection axes, in the direction of a $z_2(t)$ motion. Then the following results hold:

(i) *If $I_z > 0$, Σ_X has an unstable focus.*

(ii) *If $I_z = 0$, Σ_X is periodic.*

(iii) *If $I_z < 0$, Σ_X has a stable focus.*

Proof. When $D(\mathbb{R}^2)$ is empty

$$\dot{z}_2 = c_2 \prod_{j=1}^{l} (z_2^2 - 2z_2 a_j + a_j^2 + \beta_j^2)^{\sigma_j} x_1^{n-1}, \tag{22}$$

where $\sum_{j=1}^{l} \sigma_j = n+1$, a_j and β_j form, respectively, the real and imaginary parts of complex roots, and c_2 is a constant. But $x_1^{n-1} = x_1/(\dot{x}_1 f_1(Z^*))$, which when substituted in Eq. (22) results in

$$\int_{z_{2o}}^{z_{2f}} \frac{f_1(Z^*)\, dz_2}{c_2 \prod_{j=1}^{l} (z_2^2 - 2a_j z_2 + a_j^2 + \beta_j^2)^{\sigma_j}} = \ln x_1 \Big]_{x_{1o}}^{x_{1f}}, \tag{23}$$

where x_{1o} and x_{1f} are initial and final values of x_1 along the integral curve defined with respect to a projection variable z_2, with initial and final value z_{2o} and z_{2f}. Finally, by extending the interval (z_{2o}, z_{2f}) to cover all of the z_2 axes, the Eq. (21) is obtained.

Now, using the interpretation of Eq. (21) given in [5], together with the additional orientation information, the following is easily verified after one cycle;

(i′) $I_Z > 0$ if $x_{1f} > x_{1o}$, indicating solutions are growing.

(ii′) $I_Z = 0$ if $x_{1f} = x_{1o}$, indicating solutions are periodic.

(iii′) $I_Z < 0$ if $x_{1f} < x_{1o}$, indicating solutions are decaying.

But the statements (i′), (ii′), and (iii′) are true for any cycle, thus implying conditions (i), (ii), and (iii), respectively.

Remark. Observe that statement (ii) of Proposition 13 is almost exactly the same as Frommer's result. The only difference is that integration in Frommer's case is done without regard to a direction of the projection variable $z_2(t)$.

Propositions 12 and 13 determine asymptotic stability in the large, as well as a periodic stability for systems with n odd. Unfortunately, these properties do not exist for two-dimensional even systems. However, the next proposition is valid for Σ_X with n even or odd:

PROPOSITION 14. *The necessary and sufficient conditions for the marginal stability of Σ_X are:*

(a) *When $n = even$, then each $\lambda \in \Lambda(\mathbb{R})$ corresponding to $S \in D(\mathbb{R}^2)$ is* 0.

(b) *When $n = odd$, these two cases exist:*

(i) *If $D(\mathbb{R}^2)$ is empty, then $I_z < 0$.*

(ii) *If $D(\mathbb{R}^2)$ is nonempty, then each $\lambda \in \Lambda(\mathbb{R})$ corresponding to $S \in D(\mathbb{R}^2)$ is $\leqslant 0$.*

Proof. The proof of this proposition uses the same arguments as the proofs of Propositions 12 and 13.

REFERENCES

1. T. DATE, Classification and analysis of two-dimensional real homogeneous quadratic differential equation systems. *J. Differential Equations* **32** (1979), 311–334.
2. C. COLEMAN, "Systems of Differential Equations without Linear Terms," Sympos. Nonlinear Differential Equations and Nonlinear Mechanics, pp. 445–453, Academic Press, New York, 1963.
3. C. COLEMAN, Growth and decay estimates near non-elementary stationary points, *Canad. J. Math.* **22**, No. 6 (1970), 1156–1167.
4. H. T. DAVIS, "Introduction to Nonlinear Differential and Integral Equations," Dover, New York, 1962.
5. M. FROMMER, Über das Auftreten von Wirbeln und Strudeln (geschlossener und spiraliger Integralkurven) in der Umbgebung rationaler Unbestimmtheitsstellen, *Math. Ann.* **109** (1939), 395–424.
6. W. FULTON, "Algebraic Curves," Benjamin/Cummings, New York, 1969.
7. F. R. GANTMACHER, "The Theory of Matrices," Vol. 1, Chelsea, New York, 1977.
8. W. HAHN, "Stability of Motion," Springer-Verlag, New York, 1967.
9. J. C. HSU AND A. U. MEYER, "Modern Control Principles and Applications," McGraw–Hill, New York, 1968.
10. S. LANG, "Introduction to Algebraic Geometry," Interscience, New York, 1951.

11. L. S. LIAGHINA, The integral curves of the equations $y' = (ax^2 + bxy + cy^2)/(dx^2 + exy + fy^2)$, *Uspeki Matem. Nauk* **6**, 2(42) (1951), 171–183.
12. L. MARKUS, "Quadratic Differential Equations and Non-Associative Algebras," American Mathematical Studies 45, 185–213, Princeton Univ. Press, Princeton, New Yersey, 1960.
13. N. MINORSKY, "Nonlinear Oscillations," Van Nostrand, Princeton, New Yersey, 1962.
14. T. A. NEWTON, Two-dimensional homogeneous quadratic differential systems, *SIAM Rev.* **20**, No. 1 (1978), 120–138.
15. P. H. RABINOWITZ, Some global results for nonlinear eigenvalue problems, *J. Funct. Anal.* **7** (1971), 487–513.
16. H. RÖHRL, Finite dimensional algebras without nilpotents over algebraically closed fields, *Arch. Math.* **32** (1979), 10–12.
17. H. RÖHRL, Subalgebras that are cyclic as submodules, *Manuscripta Math.* **19** (1976), 195–209.
18. I. STAKGOLD, Branching of solutions of nonlinear equations, *SIAM Rev.* **13**, No. 3 (1971), 189–332.
19. L. K. TIMOTHY AND B. E. BONA, "State Space Analysis," McGraw–Hill, New York, 1968.
20. N. VULPE AND K. SIBIRSKII, Geometric classification of quadratic differential systems, *Differentiaľnye Uravnenija* **13**, No. 5 (1977), 548–556, (translation).